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Burr X Exponential – Weibul Distribution Properties and Application

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Abstract
In this paper, a new distribution called Burr X Exponential – Weibul distribution (BXE-W) was developed by extending weibul distribution with Burr X Exponential G family of distributions. This new distribution consists of the characteristics features of burr X, Exponential and Weibull distributions with additional four parameters. The characteristics features contain in this new distribution (BXE-W) aids to boost its flexibility. The respective density and distribution functions of this new distribution (BXE-W) were derived alongside with some mathematical properties such as moments, quantile function and order statistics. Simulation study conducted, by considering the Maximum Likelihood Estimate (MLE) method shows that the estimated parameters of BXE-W are consistent as the BIAS and RMSE approach zero. Finally, two real data sets were used to validate the results obtained from MLE method. The results show that Burr X Exponential – Weibull (BXE-W) distribution best fit the two data sets compare to the competitive distributions used in this study. Perhaps, this new distribution may be useful to model positive real life data sets that may possess the characteristics feature of Burr X, Exponential and Weibull distributions.

Keywords: Burr X Exponential – Weibull distribution; Mathematical properties; Simulation study; Application to real life data sets.

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Introduction

Modeling life time data sets with probability distributions have received great attention from researchers, especially in the field of statistical inference. These distributions have been applied in many fields such as medicine, engineering, biological science, management, and agriculture among others; among these distributions used for modeling data sets, weibull distribution is popularly known and most used by researchers.
However, weibul distribution has been used by researchers to model real life data as it improves on the flexibility of exponential distribution. This weibull distribution consists of both the shape and scale parameters while the exponential consists of only the scale parameter. Although, the complementary cumulative distributions function of a weibull distribution is a stretched exponential function. Weibull distribution is related to a number of other probability distributions; in particular, it interpolates between the exponential distribution and the Rayleigh distribution. If the quantity X is a "time-to-failure", the Weibull distribution gives a distribution for which the failure rate is proportional to a power of time, the shape parameter; is that power plus one, and so this parameter can be interpreted directly; Jiang and Murthy (2011). Thus, the Weibull distribution is not a suitable model to explain the non-monotone hazard rate function (hrf), such as

unimodal, U-shaped or bathtub form; Salman, *et. al*. (2019).

Weibull distribution has been extended by various families of distributions as aimed to improve on its flexibility in terms of capturing the non-monotone hazard rate function. These compound distributions functions are more flexible to model real life data sets, some notable ones are; Exponentiated Weibull Distribution by Pal *et. al.*, (2006), Exponentiated Kumaraswamy Exponentiated Weibull Distribution by Noor and Mundher (2018), The Exponential–Weibull lifetime distribution by Gauss *et.al.*,(2013), Poisson Burr X Weibull distribution by Abouelmagd et. al., (2019). A New Generalized Weighted Weibull Distribution by Salman *et al.* (2019), MashallOlkin Extended Weibull distribution by Ghitany *et al.* (2005), Beta Weibull distribution by Lee *et al*. (2007), the Flexible Weibull distribution by Bebbington *et al*. (2007), Kumaraswamy Weibull distribution by Cordeiro *et al*. (2010), Truncated Weibull distribution by Zhang and Xie (2011), The Topp-Leone Generated Weibull distribution by Aryal *et al*. (2016), Generalized Weibull distributions by Lai, (2014),The Kumaraswamy-Transmuted Exponentiated Modified Weibull distribution by AlBabtain *et al*. (2017), Generalized Flexible Weibull distribution by Ahmad and Iqbal (2017), A reduced New Modified Weibull distribution by Almalki (2018), and the Transmuted Exponentiated Additive Weibull distribution by Nofal *et al*. (2018).

Therefore, in this paper; a new distribution called Burr X Exponential –Weibull (BXE-W) distribution is developed by extending the Weibull distribution with Burr X Exponential – G family of distributions. The densities of this new BXE-W distribution are defined. Also, some mathematical properties are derived. Simulation study to confirm the parameters consistency of this new distribution base on the method of maximum likelihood estimate (MLE) is discussed. The flexibility of this distribution is illustrated in an application to two real life data sets. The remaining sections of the paper; is organized as follows. The cdf, pdf and density graph of the new distribution are defined in section 2. Useful expansion of BXE-W is discussed in section 3. Some mathematical properties of the BXE-W are discussed in section 4. The maximum likelihood estimate obtained for the parameters of BXE-W are presented in section 5. Simulation study with the MLE method on the efficiency of BXE-W is presented in section 6. Applications to two real data sets for the model are shown in Section 7 and Section 8 concludes the paper.

Burr X Exponential – Weibul Distribution

In this section, the cumulative distribution function (cdf) and probability density function (pdf) of the new distribution called Burr X Exponential – Weibul (BXE-W) distribution is defined.

Thus, the cdf and pdf of Burr X Exponential G family of distributions developed by Sanusi *et al.* (2020) are defined as follows:

$$F_{BXE-G}\left(x;\theta,\lambda,\beta\right)=\left[1-\exp\left\{-\left(\exp\left\{\lambda\left(\frac{G(x;\beta)}{1-G(x;\beta)}\right)\right\}-1\right)^{2}\right\}\right]^{\theta} \tag{1}$$

and

$$\begin{aligned}f_{BXE-G}\left(x;\theta,\lambda,\beta\right)=&\frac{2\theta\lambda g(x;\beta)\exp\left\{\lambda\left(\frac{G(x;\beta)}{1-G(x;\beta)}\right)\right\}}{\left[\bar{G}(x;\beta)\right]^{2}}\exp\left\{-\left(\exp\left\{\lambda\left(\frac{G(x;\beta)}{1-G(x;\beta)}\right)\right\}-1\right)^{2}\right\}\\&\times\left(\exp\left\{\lambda\left(\frac{G(x;\beta)}{1-G(x;\beta)}\right)\right\}-1\right)\left[1-\exp\left\{-\left(\exp\left\{\lambda\left(\frac{G(x;\beta)}{1-G(x;\beta)}\right)\right\}-1\right)^{2}\right\}\right]^{\theta-1}\end{aligned} \tag{2}$$

respectively.
Also, the cdf and pdf of the Weibul distribution are given as follows:

$$G(x) = 1 - \exp - \left\{ \frac{x}{\gamma} \right\}^{\phi} \tag{3}$$

and

$$g(x) = \frac{\phi}{\gamma} \left(\frac{x}{\gamma} \right)^{\phi - 1} \exp - \left\{ \frac{x}{\gamma} \right\}^{\phi} \tag{4}$$

$\vartheta, x \succ 0$, respectively.
Note: equation (5) below is considered, after the substitution of both equations (3) and (4) into (1) and (2).

$$\frac{1 - \exp - \left\{ \frac{x}{\gamma} \right\}^{\phi}}{\exp - \left\{ \frac{x}{\gamma} \right\}^{\phi}} = \exp \left\{ \frac{x}{\gamma} \right\}^{\phi} - 1 \tag{5}$$

Therefore, the cdf and pdf of the new distribution called BXE-W are respectively defined as

$$F_{BXE-W}(x; \theta, \lambda, \phi, \gamma) = \left[1 - \exp \left\{ - \left(\exp \left\{ \lambda \left(\exp \left\{ \frac{x}{\gamma} \right\}^{\phi} - 1 \right) \right\} - 1 \right)^{2} \right\} \right]^{\theta} \tag{6}$$

$$f_{BXE-W}(x; \theta, \lambda, \phi, \gamma) = \frac{2\theta\lambda \frac{\phi}{\gamma} \left(\frac{x}{\gamma} \right)^{\phi - 1} \exp \left\{ \lambda \left(\exp \left\{ \frac{x}{\gamma} \right\}^{\phi} - 1 \right) \right\}}{\exp - \left\{ \frac{x}{\gamma} \right\}^{\phi}} \exp \left\{ - \left(\exp \left\{ \lambda \left(\exp \left\{ \frac{x}{\gamma} \right\}^{\phi} - 1 \right) \right\} - 1 \right)^{2} \right\}$$
$$\times \left(\exp \left\{ \lambda \left(\exp \left\{ \frac{x}{\gamma} \right\}^{\phi} - 1 \right) \right\} - 1 \right) \left[1 - \exp \left\{ - \left(\exp \left\{ \lambda \left(\exp \left\{ \frac{x}{\gamma} \right\}^{\phi} - 1 \right) \right\} - 1 \right)^{2} \right\} \right]^{\theta - 1} \tag{7}$$

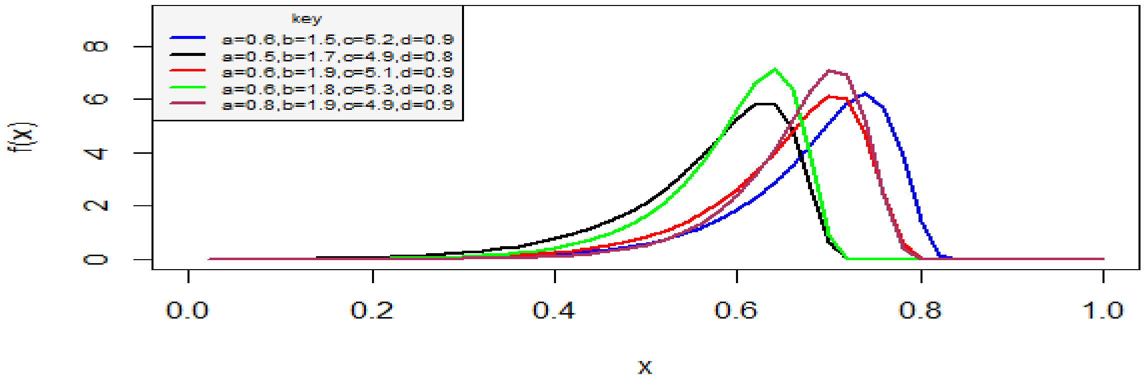

Figure 1: pd function of BXE-W

Henceforth, the survival function S(x, ζ), hazard function h(x, ζ), inverse hazard function τ (x, ζ) and cumulative hazard function H(x, ζ) for Burr X Exponential – Weibull (BXE – W) distribution are given by

$$S(x;\zeta)=1-\left[1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right]^{\theta} \tag{8}$$

$$h(x;\zeta)=\frac{2\theta\alpha\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}\left[1-\exp\left\{-\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}\right]\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}}{\left[\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}\right]^{2}\left(\exp\left\{-\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}\right)^{2}\left[1-\left[1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right]^{\theta}\right]}$$

$$\times\left[1-\exp\left\{-\left(\exp\left\{-\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right]^{\theta-1}. \tag{9}$$

$$\tau(x;\zeta)=\frac{2\theta\alpha\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}\left[1-\exp\left\{-\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}\right]\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}}{\left[\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}\right]^{2}\left(\exp\left\{-\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}\right)^{2}\left[1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right]^{\theta}}, \tag{10}$$

$$S(x;\zeta)=-In\left[1-\left[1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right]^{\theta}\right] \tag{11}$$

Important Expansion

This section introduced a useful expansion for the pdf of BXE-W. Using generalized binomial and Taylor series expansions in equation (12), thus if $|x|\prec 1$ and $k\succ 0$ is a real non integer, the power series holds:

$$(1-x)^{k-1}=\sum_{a=0}^{\infty}\frac{(-1)^{a}\Gamma(k)}{a!\Gamma(k-a)}x^{a} \tag{12}$$

Thus, applying the idea of equation (12) on the last term in (7), this becomes;

$$f_{BXE-G}\left(x;\theta,\lambda,\phi,\gamma\right)=\frac{2\theta\lambda\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}}{\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}}\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{b}$$

$$\sum_{a,b=0}^{\infty}(-1)^{a+b}\frac{a^{b}\Gamma\theta}{a!b!\Gamma(\theta-a)} \tag{13}$$

$$=\frac{2\theta\lambda\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}\right)^{1+(b-c)}}{\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}}\sum_{a,b,c=0}^{\infty}(-1)^{a+b+c}\frac{a^{b}\Gamma\theta b!}{a!b!c!\Gamma(\theta-1)(b-c)!}$$

$$=\frac{2\theta\lambda\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)^{d}}{\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}}\sum_{a,b,c,d=0}^{\infty}(-1)^{a+b+c}\frac{a^{b}\Gamma\theta b!\left(1+(b-c)\right)^{d}\lambda^{d}}{a!b!c!d!\Gamma(\theta-1)(b-c)!} \tag{14}$$

$$=2\theta\lambda\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\right)^{d-e}\sum_{a,b,c,d,e=0}^{\infty}(-1)^{a+b+c+e}\frac{a^{b}\Gamma\theta b!d!\left(1+(b-c)\right)^{d}\lambda^{d}}{a!b!c!d!e!\Gamma(\theta-1)(b-c)!(d-e)!}\left[\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}\right]^{-1}$$

$$=2\theta\lambda\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\exp\left\{(d-e)\left(\frac{x}{\gamma}\right)^{\phi}\right\}\sum_{a,b,c,d,e=0}^{\infty}(-1)^{a+b+c+e}\frac{a^{b}\Gamma\theta b!d!\left(1+(b-c)\right)^{d}\lambda^{d}}{a!b!c!d!e!\Gamma(\theta-1)(b-c)!(d-e)!}\exp\left\{\left(\frac{x}{\gamma}\right)^{\phi}\right\}$$

$$=2\theta\lambda\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\sum_{a,b,c,d,e=0}^{\infty}(-1)^{a+b+c+e}\frac{a^{b}\Gamma\theta b!d!\left(1+(b-c)\right)^{d}\lambda^{d}}{a!b!c!d!e!\Gamma(\theta-1)(b-c)!(d-e)!}\exp\left\{\left[1+(d-e)\right]\left(\frac{x}{\gamma}\right)^{\phi}\right\} \tag{15}$$

$$=\sum_{a,b,c,d,e=0}^{\infty}\Omega_{a,b,c,d,e}\frac{x^{\gamma-1}}{\gamma^{\phi}}\exp\left\{\left[1+(d-e)\right]\left(\frac{x}{\gamma}\right)^{\phi}\right\}\quad where\;\;\Omega_{a,b,c,d,e}=(-1)^{a+b+c+e}\frac{2\theta\lambda^{d+1}\phi a^{b}\Gamma\theta b!d!\left(1+(b-c)\right)^{d}}{a!b!c!d!e!\Gamma(\theta-1)(b-c)!(d-e)!}$$

Mathematical Properties

This section provides some mathematical properties of the BXE-W distribution such as moments, quantile function and order statistics.

Moments

Suppose X is a random variable with BXE-W distribution, then the raw moment, say $\mu_n^{'}$, is given by

$$\mu_n^{'}=E\left(x^{n}\right)=\int_{-\infty}^{\infty}x^{n}f_{BXE-W}\left(x;\sigma,\lambda,\phi,\gamma\right)dx \tag{16}$$

$$\mu_n^{'}=\sum_{a,b,c,d,e=0}^{\infty}\Omega_{a,b,c,d,e}\frac{x^{n+(\gamma-1)}}{\gamma^{\phi}}\int_{-\infty}^{\infty}\exp\left\{\left[1+(d-e)\right]\left(\frac{x}{\gamma}\right)^{\varphi}\right\}dx$$

$$\Omega_{a,b,c,d,e} = (-1)^{a+b+c+e} \frac{2\theta\lambda^{d+1}\varphi a^{b}\Gamma\theta b!d!\left(1+(b-c)\right)^{d}}{a!b!c!d!e!\Gamma(\theta-1)(b-c)!(d-e)!}dx$$

Quantile function

The quantile function of BXE-W is derived as:

$$\text{Thus,} \quad x \;\Rightarrow\; Q(u) \;\Rightarrow\; 1-\exp-\left\{\frac{x}{\gamma}\right\}^{\phi} = \frac{k}{(1+k)}$$

$$\Rightarrow 1-\frac{k}{(1+k)} = \exp-\left\{\frac{x}{\gamma}\right\}^{\phi} \qquad \Rightarrow -\log\left(1-\frac{k}{(1+k)}\right) = \left\{\frac{x}{\gamma}\right\}^{\phi}$$

$$\Rightarrow -\left(\log\left(1-\frac{k}{(1+k)}\right)\right)^{\frac{1}{\phi}} = \frac{x}{\gamma} \qquad \Rightarrow \text{Therefore,} \quad x = \frac{\left(\log\left(1-\frac{k}{(1+k)}\right)\right)^{\frac{1}{\phi}}}{-\gamma}$$

$$\text{where} \quad k = \frac{\left(\log\left[\left(-\log\left(1-u^{\frac{1}{\theta}}\right)\right)^{\frac{1}{2}}+1\right]\right)}{\lambda}$$

Order Statistics

$$f_{i.n} = \frac{f(x)}{\beta(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}F^{j+i-1}(x)$$

$$f_{i.n} = \frac{1}{\beta(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}f(x)F^{j+i-1}(x)$$

$$f(x)F^{j+i-1}(x) =$$

$$= \frac{2\theta\lambda\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}}{\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}}\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}$$

$$\times\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)\left[1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right]^{\theta(1+(j+i-1))-1}$$

$$= \frac{2\theta\lambda\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}}{\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}}\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2a+1}$$

$$\times\sum_{a,b=0}^{\infty}(-1)^{a+b}\frac{(a+1)^{b}\,\Gamma\left[\theta\left(1+(j+i-1)\right)\right]}{a!b!\Gamma\left[\theta\left(1+(j+i-1)\right)-a\right]}$$

$$=\frac{2\theta\lambda\frac{\phi}{\gamma}\left(\frac{x}{\gamma}\right)^{\phi-1}}{\exp-\left\{\frac{x}{\gamma}\right\}^{\phi}}\sum_{a,b,c,d,e=0}^{\infty}(-1)^{a+b+c+e}\frac{(a+1)^{b}\,\Gamma\left[\theta\left(1+(j+i-1)\right)\right]}{a!b!c!d!e!}$$

$$\times\frac{(2a+1)!\left(\left[(2a+1)-c\right]+1\right)^{d}\lambda^{d}d!}{\Gamma\left[\theta\left(1+(j+i-1)\right)-a\right]\left[(2a+1)-c\right]!(d-e)!}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\right)^{d-e}$$

$$=\sum_{a,b,c,d,e,f=0}^{\infty}(-1)^{a+b+c+e}\frac{2\theta\phi(a+1)^{b}\,\Gamma\left[\theta\left(1+(j+i-1)\right)\right](2a+1)!\left(\left[(2a+1)-c\right]+1\right)^{d}}{a!b!c!d!e!f!\Gamma\left[\theta\left(1+(j+i-1)\right)-a\right]}$$

$$\times\frac{\lambda^{d+1}d!(d-e)^{f}}{\left[(2a+1)-c\right]!}\frac{x^{\phi(1+f)-1}}{\gamma^{\phi(1+f)}}\exp\left\{\frac{x}{\gamma}\right\}^{\phi}$$

$$f_{i,n}=\sum_{j=0}^{n-i}(-1)^{j}\sum_{a,b,c,d,e=0}^{\infty}(-1)^{a+b+c+e+j}\frac{2\theta\varphi(a+1)^{b}\,\Gamma\left[\theta\left(1+(j+i-1)\right)\right](2a+1)!\left(\left[(2a+1)-c\right]+1\right)^{d}}{a!b!c!d!e!f!\Gamma\left[\theta\left(1+(j+i-1)\right)-a\right]}$$

$$\times\frac{1}{\beta(i,n-i+1)}\frac{\lambda^{d+1}d!(d-e)^{f}}{\left[(2a+1)-c\right]!}\binom{n-i}{j}\frac{x^{\phi(1+f)-1}}{\gamma^{\phi(1+f)}}\exp\left\{\frac{x}{\gamma}\right\}^{\phi}$$

Parameter Estimation

Several approaches are used to estimate a parameter, but the maximum likelihood method is the most commonly used among others. Therefore, the maximum likelihood estimators of the unknown parameters of the BXE-W distribution from complete samples are determined. Let $X_1,\ldots,X_n$ be observed values from the BXE-W distribution with a vector of parameters ϕ. The Log-likelihood function can be expressed as

$$l(\phi)=n\log(2)+n\log(\theta)+n\log(\lambda)+n\log(\phi)-n\log(\gamma)+(\phi-1)\sum_{i=1}^{n}\log\left(\frac{x}{\gamma}\right)+\sum_{i=1}^{n}\left(\frac{x}{\gamma}\right)^{\phi}$$

$$+\lambda\sum_{i=1}^{n}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)-\sum_{i=1}^{n}\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}+\lambda\sum_{i=1}^{n}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)$$

$$-n\log(1)+(\theta-1)\sum_{i=1}^{n}\log\left(1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right)$$

Differentiating equation above with respect to $\theta,\lambda,\varphi,\gamma$ respectively and equating them to 0, we have:

$$\frac{n}{\theta}+\sum_{i=1}^{n}\log\left(1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right)\quad=\quad 0$$

$$\frac{n}{\lambda}+\sum_{i=1}^{n}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)-\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}\sum_{i=1}^{n}\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}$$

$$+\sum_{i=1}^{n}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)+(\theta-1)\sum_{i=1}^{n}\frac{\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}}{\left(1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right)}$$

$$\times\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}=0$$

$$\frac{n}{\phi}+\sum_{i=1}^{n}\log\left(\frac{x}{\gamma}\right)+\sum_{i=1}^{n}\left(\frac{x}{\gamma}\right)+\lambda\left(\frac{x}{\gamma}\right)\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\sum_{i=1}^{n}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)-\left(\frac{x}{\gamma}\right)\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}$$
$$\times\sum_{i=1}^{n}\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}+\lambda\left(\frac{x}{\gamma}\right)\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\sum_{i=1}^{n}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{\left(\frac{x}{\gamma}\right)\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}}{\left(1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right)}\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}=0$$

$$-\frac{n}{\gamma}-(\phi-1)\sum_{i=1}^{n}\left(\frac{1}{\gamma}\right)+\sum_{i=1}^{n}x^{\phi}+\lambda x^{\phi}\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\sum_{i=1}^{n}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)-x^{\phi}\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}$$
$$\times\sum_{i=1}^{n}\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}+\lambda x^{\phi}\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\sum_{i=1}^{n}\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{x^{\phi}\exp\left\{\frac{x}{\gamma}\right\}^{\phi}\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}}{\left(1-\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}\right)}\exp\left\{-\left(\exp\left\{\lambda\left(\exp\left\{\frac{x}{\gamma}\right\}^{\phi}-1\right)\right\}-1\right)^{2}\right\}=0$$

Simulation Study

In this section, simulation study for different sample size on the efficiency of MLE method to determine the parameters' consistency of BXE-W is carried out. Thus, a random variable X from BXE-W is generated using R Software. Samples of size n = 10; 20; 40; 60; 80 and 100 from BXE-W distribution for some selected combination of parameters are used. This process is repeated N = 1000 time to calculate mean estimate, means squared error and bias. The results obtained are given in table 6.1 below.

From Table 6.1, it is observed that when sample size increases the mean of the MLE approaches the initial values. Bias and Root Mean Squared Error (RMSE) decreases towards zero, that is the parameters of BXE-W distribution are consistent. Therefore, the maximum likelihood method works very well to estimate the parameters of BXE-W distribution.

Table 6.1: Means, Bias and RMSEs for the BXE-W parameter estimates when $\theta=0.7,\lambda=0.4,\phi=0.6,\gamma=0.3$

		MLE			
N		θ	λ	ϕ	γ
10	**Mean**	0.7006	0.3718	0.8001	0.2885
	Bias	0.0006	-0.0282	0.2001	-0.0115
	RMSE	0.3206	0.1115	0.3739	0.0646
20	**Mean**	0.7026	0.3946	0.7205	0.3012
	Bias	0.0026	-0.0054	0.1205	0.0012
	RMSE	0.2760	0.0833	0.2691	0.0475
40	**Mean**	0.7049	0.4046	0.6699	0.3065
	Bias	0.0049	0.0046	0.0699	0.0065
	RMSE	0.2376	0.0531	0.1994	0.0361
60	**Mean**	0.7038	0.4072	0.6542	0.3084
	Bias	0.0038	0.0072	0.0542	0.0084
	RMSE	0.2050	0.0451	0.1711	0.0310

80	**Mean**	0.7082	0.4077	0.6406	0.3077
	Bias	0.0082	0.0077	0.0406	0.0077
	RMSE	0.1929	0.0400	0.1497	0.0281
100	**Mean**	0.7094	0.4086	0.6322	0.3077
	Bias	0.0094	0.0086	0.0322	0.0077
	RMSE	0.1690	0.0346	0.1297	0.0245

Application to Data Sets

The flexibility of BXE-W in application to real life data sets is illustrated by comparing its performance with some other existing distributions. The parameters are estimated using maximum likelihood method. The goodness-of-fit statistic and the MLE's for the models' parameters are presented in Table 7.1. To compare the fitted models, this paper used a goodness-of-fit measure which is Akaike information criterion (AIC).

The fits of this new BXE-W distribution with other competitive distributions such as Topp Leone Exponential Weibull (TLE-W), Transmuted Weibull (TW), Burr X Exponential Lomax (BXE-L) Exponentiated Generalized Weibull (ETG-W) distributions are compared. Their PDFs are available in literature.

Throughout the results presented in the two tables, the distribution with the lowest AIC values is BXE-W distribution compares to other competitive distributions; that is, BXE-W best fits the two data sets.

Thus, the two data sets considered are:

Data Set 1

The first data represents the remission times (in months) of a random sample of 128 bladder cancer patients previously used by Lee and Wang (2003) and A. A Sanusi*et. al.* (2020). Below is the data set:

0.08, 5.85, 8.26, 11.98, 19.13, 1.76, 10.34, 14.83, 3.88, 5.32, 7.39, 3.25, 4.50, 2.09, 3.48, 4.87, 0.81, 2.62,11.64, 17.36, 1.40, 3.02, 4.34, 34.26, 0.90 , 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.33, 5.49, 7.66, 11.25, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 17.14, 79.05, 1.35, 2.87, 12.07, 21.73, 2.07, 3.36, 6.93,5.62, 7.87, 3.82, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 14.77, 32.15, 2.64, 5.71, 7.28, 9.74, 14.76, 26.31, 5.32, 7.32, 10.06, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69,4.23, 7.93, 11.79, 18.10, 1.46, 4.40, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 8.65, 12.63, 22.69.

Data Set 2

The second data set represents the survival times of 121 patient with breast cancer obtained from a large hospital in a period from 1929 – 1938 previously used by Lee (1992), Ramos *et al.*, (2013) and A. A Sanusi *et. al.* (2020).Below is the data set:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0

In Tables 7.1 and 7.2, the values of log-likelihood (LL) and AIC are minimum and favorable of BXE-W distribution than other existing distributions, which indicates that the new model (BXE-W) best fit the sets of data. It is depicted from the results that our proposed model provides best fit compare to other sub models. That is, it is more reliable with this type of data set. It is also clear that the BXE-W distribution provides the best fit as compare to TLE-W, TW, W, BXE-L and ETG-W for the given two data sets. So, the BXE-W model could be chosen as the best model.

Table 7.1. Parameters Estimation for Various Distributions depending on data set 1.

	Models	Estimates	$-LL$	AIC	Rank
MLE	BXE-W	θ =6.4605 λ =0.4750 ϕ =0.0929 γ =2.8168	410.754	829.508	1
	TLE-W	σ = 7.6289 λ =0.6112 ϕ =0.2638 γ =0.4381	410.933 829.8660		2
	TW	ϑ =1.1337 ϕ =14.6183 γ =0.7440	411.958	829.9168	3
	W	ϕ =1.0477 γ =9.5613	414.0 87	832.1738	4
	BXE-L	θ = 0.4978 λ =12.9334 φ =0.0295 α =3.2198	424.268	856.5362	5
	ETG-W	δ =1.0537 ψ =1.3702 ϕ =0.2765 γ =0.4993	540.279	1088.559	6

Table 7.2. Parameters Estimation for Various Distributions depending on data set 2.

	Models	Estimates	$-LL$	AIC	Rank
MLE	BXE-W	θ =2.0283 λ =0.2480 ϕ =0.1688 γ =5.4315	578.6577	1165.315	1
	TLE-W	σ =2.4973 λ =0.0435 ϕ =0.2686 γ	578.66	1165.32	2
	TW	ϑ =0.4785 ϕ =11.0835 γ =-1.4375	603.3242	1212.648	3
	BXE-L	θ =8.3370 λ =28.9403 φ =0.0119 α =0.0041	606.8674	1221.735	4
	ETG-W	δ =0.8950 ψ =1.2991 ϕ =0.1684 γ =0.2231	772.4545	1552.909	5

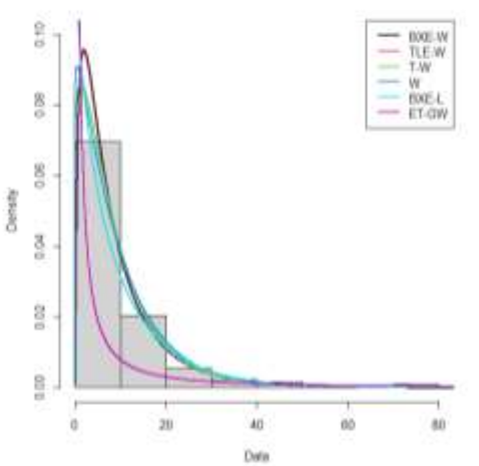

Figure 1: Histogram of Data Set 1

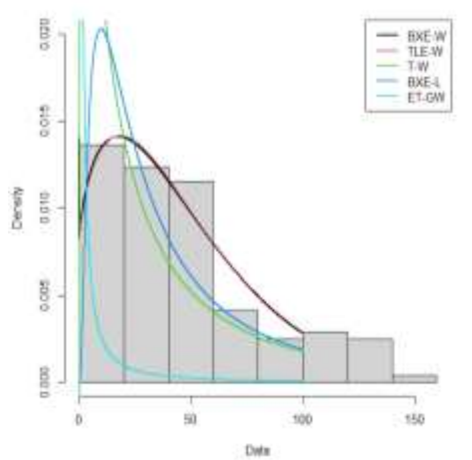

Figure 2: Histogram of Data Set 2

Conclusion

Weibul distribution is extended with Burr X Exponential G family of distributions and resulted to Burr X Exponential Weibull (BXE-W) distribution with an additional four parameters. Some relatively mathematical properties of this compound distribution are computed base on the Binomial theorem expansion. The efficiency performance of the four parameters is proved to be consistent, however tested through simulation study with the method of maximum likelihood Estimate showed that the parameters are unbiased. Finally, this new BXE-W distribution is applied to real data sets to assess its flexibility over the existing distributions. It is significantly observed that the new distribution best fit the data sets compare to the other distributions used in this study.

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