Jewel Journal of Scientific Research (JJSR) 3(1): 73– 78, 2015
©Federal University, Kashere-Nigeria (ISSN: 2384 - 6267)
www.fukashere.edu.ng

On generalized 4-step implicit linear multistep method for the numerical solution of second order ordinary differential equations

Ibrahim, S., Rauf, K. and Aniki, A. S.
Department of Mathematics,
University of Ilorin, Ilorin, Nigeria.
Author for Correspondence: krauf@unilorin.edu.ng

Abstract
In this article, the numerical solutions of Initial Value Problems (IVPs) of general second order Ordinary Differential Equations have been studied. Continuous implicit block 4 step method for solving this type of problem has been developed through the idea of Taylor's expansion. The main scheme obtained was implemented together with the block formula for the numerical solutions of second order ordinary differential equations. Numerical example was presented to illustrate the applicability and efficiency of the method. The result obtained when compared with exact solution and existing method are favourable.

Key words: General second order, IVPs, Block method, Self starting.

Introduction

Many fields of applications, notably in Mathematics, Sciences and Engineering yield Initial Value Problems of Second order ordinary differential equations are of the form:

$$y'' = f\left(x, y, y'\right), y(a) = y, y'(a) = \beta \quad (1)$$

Many of such problems may not be easily solved analytically, hence numerical schemes are developed to solve such equations. These problems are always reduced to systems of first order equations and numerical methods of first order differential equations are used to solve them (Awoyemi,1999). This approach is inefficient; a lot of human efforts were wasted on implementation due to the nature of problems (Butcher, 2003).

Some researchers have attempted the solution of (1) directly without reduction to a first order system of equations, Chu and Hamilton (1987) proposed a generalization of the linear multistep method to a class of multi-block methods to solve second order initial value problems type (1) directly, see also Simo (2002). In a resent paper of Jator and Li (2009), a proposed direct block method as a self starting method for accurate approximation to y' appearing in equation (1) to be able to solve problem(1) directly was made, see also Fatunla (1994). The aim of this paper is to demonstrate by using the present block method of order $(6,6,6)^T$derived to solve equation (1) directly and compare its performance with the predictor – corrector method proposed by Awoyemi (2005) and Kayode (2004).

Analysis of the method

The numerical method of consideration for direct integration of general second order differential equation of type (1) is of the form:

$$\sum_{j=0}^{K} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{K} \beta_j f_{n+j} \quad (2)$$

which is expressed as:

$$y_{n+k} = \alpha_0 y_n + \alpha_1 y_{n+1} + \cdots + \alpha_{k-1} y_{n+k-1} + h^2[\beta_0 f_n + \beta_1 f_{n+1} \dots + \beta_k f_{n+k}$$
]

where y_{n+j} is an approximation to $y(x_{n-j})$ and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$

The co-efficient α_j and β_j are constants which do not depend on $n$subject to the conditions $\alpha_k = 1, |\alpha_0| + |\beta_0| \neq 0$ and are determined to ensure that the method is consistent and zero stable. Also, method (2) is implicit since $\beta_k \neq 0$, Ismail (2009) and can be written symbolically as:

$\rho(E)\, y_n - h^{\mu}\delta(E) = 0, f_n = f(x_n, y_n)$

where E is the shift operator defined by $E^i y_n = y_{n+j}$ while $\rho(E)$ and $\delta(E)$ are respectively the first and second characteristic polynomial of the linear multistep method as:

$$\rho(E) = \sum_{j=0}^{K} \alpha_j E^j, \alpha_k \neq 0\,, \qquad \delta(E) = \sum_{j=0}^{k} \beta_j E^j$$

Following Fatunla (1988) and Lambert (1973), we define the local truncation error associated with equation (2) by the difference operator

$$L[y(x):h] = \sum_{j=0}^{k} [\alpha_j\, y(x_n + jh) - h^2 \beta_j y''(x_n + jh)] \qquad (3)$$

Where $y(x)$ is assumed to have continuous derivatives of sufficiently high order. Therefore, expanding equation (3) in Taylor's series about the point x to obtain the following expression:

$$L[y(x):h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \cdots + c_q h^2 y^q(x) + \cdots \qquad (4)$$

where:

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k$$

$$c_1 = \alpha_1 + 2\,\alpha_2 + \cdots + k\,\alpha_k$$

$$c_2 = \frac{1}{2!}[\alpha_1 + 2^2\,\alpha_2 + \cdots k^2\,\alpha_k] - [\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k]$$

$$\vdots$$

$$c_q = \frac{1}{q!}[\alpha_1 + 2^q\,\alpha_2 + \cdots k^q\,\alpha_k] - \frac{1}{(q-2)!}[\beta_1 + 2^{q-2}\beta_2 + \cdots + k^{q-2}\beta_k]$$

In the sense of Lambert (1973), we say that, the method has order q if,

$c_0 = c_1 = \cdots = c_q = c_{q+1} = 0$ and $c_{q+2} \neq 0$.

c_{q+2} is called error constant and implies that the truncation error at the point x_n is given by

$$T_{n+k} = c_{q+2} h^{q+2} y^{(q+2)}(x_n) + 0(h^{q+3})$$

Derivation of the schemes

We construct an implicit linear four step method of maximal order two, containing one free parameter. That is:

$$\sum_{j=0}^{4} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{4} \beta_j f_{n+j} \qquad (5)$$

This can be expressed as:

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} = h^2[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4}] \qquad (6)$$

$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0, \quad c_8 \neq 0$

Since $k = 4$ then, $\alpha_4 = 1$

By hypothesis, let $\alpha_0 = b$, where b is the free parameter

The remaining unknown parameters are: $\alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3$, and β_4

We expand the equation (6) in Taylor's series as:

$$\alpha_0 y_n = \alpha_0 y_n \qquad (7)$$

$$\alpha_1 y_{n+1} = \alpha_1 y_n [x_n + h] = \alpha_1 [y_n + h y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \frac{h^4}{4!} y^{iv}_n + \frac{h^5}{5!} y^{v}_n + \cdots] \qquad (8)$$

$$\alpha_2 y_{n+2} = \alpha_2 y_n [x_n + 2h] = \alpha_2 [y_n + 2h\, y'_n + \frac{(2h)^2}{2!} y''_n + \frac{(2h)^3}{3!} y'''_n + \frac{(2h)^4}{4!} y^{iv}_n + \frac{(2h)^5}{5!} y^{v}_n + \cdots] \qquad (9)$$

$$\alpha_3 y_{n+3} = \alpha_3 y_n [x_n + 3h] = \alpha_3 [y_n + 3h\, y'_n + \frac{(3h)^2}{2!} y''_n + \frac{(3h)^3}{3!} y'''_n + \frac{(3h)^4}{4!} y^{iv}_n + \frac{(3h)^5}{5!} y^{v}_n + \cdots] \qquad (10)$$

$$\alpha_4 y_{n+4} = \alpha_4 [x_n + 4h] = \alpha_4 [y_n + 4h\, y'_n + \frac{(4h)^2}{2!} + y_n + \frac{(4h)^3}{3!} y'_n + \frac{(4h)^4}{4!} y^{iv}_n + \frac{(4h)^5}{5!} y^{v}_n + \cdots] \qquad (11)$$

$\beta_0 f_n = \beta_0 y'_n$ (12)

$\beta_1 f_{n+1} = \beta_1 y'_n[x_n + h]$

$= \beta_1[y'_n + hy''_n + \frac{h^2}{2!}y'''_n + \frac{h^3}{3!}y^{iv}_n + \frac{h^4}{4!}y^v_n + \frac{h^5}{5!}y^{vi}_n + \cdots]$

(13)

$\beta_2 f_{n+2} = \beta_1 y'_n[x_n + 2h]$

$= \beta_2[y'_n + 2hy''_n + \frac{(2h)^2}{2!}y'''_n + \frac{(2h)^3}{3!}y^{iv}_n + \frac{(2h)^4}{4!}y^v_n + \frac{(2h)^5}{5!}y^{vi}_n + \cdots]$

(14)

$\beta_3 f_{n+3} = \beta_3 y'_n[x_n + 3h]$

$= \beta_3[y'_n + 3hy''_n + \frac{(3h)^2}{2!}y'''_n + \frac{(3h)^3}{3!}y^{iv}_n + \frac{(3h)^4}{4!}y^v_n + \frac{(3h)^5}{5!}y^{vi}_n + \cdots]$

(15)

$\beta_4 f_{n+4} = \beta_4 y'_n[x_n + 4h]$

$= \beta_4[y'_n + 4hy''_n + \frac{(4h)^2}{2!}y'''_n + \frac{(4h)^3}{3!}y^{iv}_n + \frac{(4h)^4}{4!}y^v_n + \frac{(4h)^5}{5!}y^{vi}_n + \cdots]$

(16)

By comparing the series (7) to (16) with (4), we have;

$c_o = \propto_0 + \propto_1 + \propto_2 + \propto_3 + \propto_4 = 0$ (17)

$c_1 = \propto_1 + 2\propto_2 + 3\propto_3 + 4\propto_4 = 0$ (18)

$c_2 = \frac{1}{2!}[\propto_1 + 2^2\propto_2 + 3^2\propto_3 + 4^2\propto_4] - [\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4] = 0$

(19)

$c_3 = \frac{1}{3!}[\propto_1 + 2^3\propto_2 + 3^3\propto_3 + 4^3\propto_4] - [\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4] = 0$

(20)

$c_4 = \frac{1}{4!}[\propto_1 + 2^4\propto_2 + 3^4\propto_3 + 4^4\propto_4] - \frac{1}{2!}[\beta_1 + 2^2\beta_2 + 3^2\beta_3 + 4^2\beta_4] = 0$

(21)

$c_5 = \frac{1}{5!}[\propto_1 + 2^5\propto_2 + 3^5\propto_3 + 4^5\propto_4] - \frac{1}{3!}[\beta_1 + 2^3\beta_2 + 3^3\beta_3 + 4^3\beta_4] = 0$

(22)

$c_6 = \frac{1}{6!}[\propto_1 + 2^6\propto_2 + 3^6\propto_3 + 4^6\propto_4] - \frac{1}{4!}[\beta_1 + 2^4\beta_2 + 3^4\beta_3 + 4^4\beta_4] = 0$

(23)

$c_7 = \frac{1}{7!}[\propto_1 + 2^7\propto_2 + 3^7\propto_3 + 4^7\propto_4] - \frac{1}{5!}[\beta_1 + 2^5\beta_2 + 3^5\beta_3 + 4^5\beta_4] = 0$

(24)

$c_8 = \frac{1}{8!}[\propto_1 + 2^8\propto_2 + 3^8\propto_3 + 4^8\propto_4] - \frac{1}{6!}[\beta_1 + 2^6\beta_2 + 3^6\beta_3 + 4^6\beta_4] \neq 0$

By substituting $\propto_0 = b, \propto_4 = 1$ in equations (17), (18) and simplifying the equations (19) to (24), we have:

$\propto_1 + \propto_2 + \propto_3 = -1 - b$ (25)

$\propto_1 + 2\propto_2 + 3\propto_3 = -4$ (26)

$\propto_1 + 4\propto_2 + 9\propto_3 - 2[\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4] = -16$

(27)

$\propto_1 + 8\propto_2 + 27\propto_3 - 6[\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4] = -64$

(28)

$\propto_1 + 16\propto_2 + 81\propto_3 - 12[\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4] = -256$

(29)

$\propto_1 + 32\propto_2 + 243\propto_3 - 20[\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4] = -1024$

(30)

$\propto_1 + 64\propto_2 + 729\propto_3 - 30[\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4] = -4096$

(31)

$\propto_1 + 128\propto_2 + 2187\propto_3 - 42[\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4] = 0$

(32)

From equations (25) and (26), we have:

$\propto_2 = 2\propto_3 - 3 + b$

$\propto_1 = \propto_3 + 2 - 2b$

By substituting the values of $\propto_1$ and $\propto_2$ in equations (28) to (32), we obtain the following equations:

$12\propto_3 - 6[\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4] = -42 - 6b$

(33)

$50\propto_3 - 12[\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4] = -210 - 14b$

(34)

$180\propto_3 - 20[\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4] = -930 - 30b$

(35)

$$602 \propto_3 - 30[\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4] = -3906 - 62b \quad (36)$$

$$1932 \propto_3 - 42[\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4] = -16002 - 126b \quad (37)$$

From the equation (33), we have:

$$\propto_3 = \frac{1}{2}[\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 - 7 - b]$$

Substitute the value of $\propto_3$into equations (33) to (37) to obtain:

$$13\beta_1 + 2\beta_2 - 33\beta_3 - 92\beta_4 = -35 + 11b \quad (38)$$

$$60\beta_1 + 20\beta_2 - 270\beta_3 - 920\beta_4 = -300 + 60b \quad (39)$$

$$271\beta_1 + 122\beta_2 - 1527\beta_3 - 6476\beta_4 = -1799 + 239b \quad (40)$$

$$924\beta_1 + 588\beta_2 - 7308\beta_3 - 39144\beta_4 = -9240 + 840b \quad (41)$$

From equation (38), we have:

$$\beta_2 = \frac{1}{2}[33\beta_3 - 13\beta_1 + 92\beta_4 - 35 + 11b]$$

Substitute the value of β_2 into equations (39) to(41) to get:

$$-70\beta_1 + 60\beta_3 = 50 - 50b \quad (42)$$

$$-522\beta_1 + 486\beta_3 - 864\beta_4 = 336 - 432b \quad (43)$$

$$-2898\beta_1 + 2394\beta_3 - 12096\beta_4 = 1050 - 2394b \quad (44)$$

From the equation (42), we have:

$$\beta_3 = \frac{1}{60}[70\beta_1 + 50 - 50b]$$

Substitute the value of β_3 into equations (43) and (44) to obtain:

$$15\beta_1 - 288\beta_4 = -23 - 9b \quad (45)$$

$$-35\beta_1 - 4032\beta_4 = -315 - 133b \quad (46)$$

By solving the equations (45) and (46), we have:

$$\beta_1 = \frac{1}{735}[1066 + 462b], \qquad \beta_4 = \frac{1}{14112}[1066 + 462b]$$

Substitute the value of β_1 into equation (46) to get:

$$\beta_3 = \frac{1591 - 63b}{630}$$

Substitute the value of β_1, β_3 and β_4 into equation (38) to obtain:

$$\beta_2 = \frac{63004 - 74613b}{17640}$$

By substituting the value of $\beta_1, \beta_2, \beta_3$ and β_4 into equation (33) and putting the result obtained in equations (25) and (26), we have:

$$\propto_3 = \frac{145283 - 70560b}{17640}, \propto_2 = \frac{-171743 - 79380b}{8820}$$

and $\propto_1 = \frac{180563 - 105840b)}{17640}$

Substitute the values of $\propto_1, \propto_2, \propto_3, \beta_0, \beta_1, \beta_2, \beta_3$ and β_4 into equation (27) to get:

$$\beta_0 = \frac{127469 + 23583b}{35280}$$

By putting the values of $\propto_0, \propto_1, \propto_2, \propto_3, \propto_4, \beta_0, \beta_1, \beta_2, \beta_3$, and β_4 into equation (6), we obtain the following schemes:

$$by_n + \frac{(180563 - 105840b)}{17640} y_{n+1} + \frac{(-171743 + 79380b)}{8820} y_{n+2} + \frac{(145283 - 70560b)}{17640} y_{n+3} + y_{n+4} = \frac{h^2}{35280}\left[(12746 + 23583b) f_n + 4(1066 - 462b) f_{n+1} + 2(63004 - 74613b) f_{n+2} + 56(1591 - 63b) f_{n+3} + 5(533 + 231b) f_{n+4}\right] \quad (47)$$

Since b is a free parameter,

Let $b = -3, -2, 2$ and 3, and substitute into equation (47) to obtain the following proposed block scheme:

$$-3y_n + \frac{498083}{17640} y_{n+1} - \frac{409883}{8820} y_{n+2} + \frac{35693}{17640} y_{n+3} + y_{n+4} = \frac{h^2}{35280}[5800 f_n - 15360 f_{n+1} + 573686 f_{n+2} + 99680 f_{n+3} - 800 f_{n+4}]$$

$$-2y_n + \frac{392243}{17640} y_{n+1} - \frac{330503}{8820} y_{n+2} + \frac{286403}{17640} y_{n+3} + y_{n+4} = \frac{h^2}{35280}[-34420 f_n + 6816 f_{n+1} + 424460 f_{n+2} + 96152 f_{n+3} + 355 f_{n+4}]$$

$$2y_n - \frac{31117}{17640} y_{n+1} - \frac{12983}{8820} y_{n+2} + \frac{4163}{17640} y_{n+3} + y_{n+4} = \frac{h^2}{35280}[59912 f_n + 95520 f_{n+1} - 172444 f_{n+2} + 82040 f_{n+3} + 4975 f_{n+4}]$$

$$3y_n - \frac{136957}{17640}y_{n+1} + \frac{66397}{8820}y_{n+2} + \frac{66397}{17640}y_{n+3} + y_{n+4} = \frac{h^2}{35280}[83495f_n + 117696f_{n+1} - 321670f_{n+2} + 78512f_{n+3} + 6130f_{n+4}] \quad (48)$$

The block scheme of 48 is of orders $[6,6,6,6]^T$ with error constants $[\frac{26}{889}, \frac{415}{1776}, -\frac{335}{1776}, -\frac{54}{889}]^T$

Numerical experiment

In order to confirm the efficiency of the proposed method for $k = 4$, The following Initial Value Problem was used as test problem:

$$y'' - x(y')^2 = 0, y(0) = 1 = y'(0) = \frac{1}{2}, h = 0.05$$

Exact solution: $y(x) = 1 + \frac{1}{2}\ln\frac{2+x}{2-x}$

Source: Awoyemi (2005) and Kayode (2004)

Our results were compared with the exact solution and results of Awoyemi (2005) and Kayode (2004). All calculations and program are carried out with the aid of Maple Software. The approximate solution for this problem using the proposed method is presented in the below:

Table 1:Comparing the errors in the proposed methods to errors in Awoyemi and Kayode.

X	Exact Solution	Approximate Solution	Error in Proposed Method	Error in Awoyemi [2005]	Error in Kayode [2004]
0.1	1.05004172927849140	1.0500417292784030	1.1102E–15	6.6391E–14	7.1629E–12
0.2	1.10033534773107560	1.1003353477310696	5.9952E–15	2.0012E–10	1.5091E–11
0.3	1.15114043593646700	1.1511404359356441	2.5535E–14	1.7200E–09	4.5286E–11
0.4	1.20273255405936467	1.2027325540540112	7.1054E–14	5.8946E–09	1.0808E–10
0.5	1.25541281188299550	1.2554128118828796	1.1568E–13	1.4434E–08	1.7818E–10
0.6	1.30654437542713964	1.3095196041202992	1.1990E–13	4.1866E–08	4.4434E–10
0.7	1.36544375427139640	1.3654437542720822	6.8567E–13	5.3109E–08	7.4446E–10
0.8	1.42364893019360220	1.4236489301970776	3.4754E–12	9.1131E–08	1.5009E–10
0.9	1.48470027859405200	1.4847002786062715	1.2219E–11	1.4924E–07	3.7579E–09
1.0	1.54930614433405540	1.5493061443713383	3.7282E–11	2.3718E–07	4.7410E–09

Conclusion

The constructed k-step linear multistep method using Taylor series as a basis function for approximate solution was used. A new scheme with continuous coefficient is obtained which was applied to solve special and general second-order Initial Value Problem in ordinary differential equation. Evidence of the better accuracy of our method over existing methods is as shown in Table above.

References

Awoyemi, D. O. (1999). "A New Sith-oder Algorithm for General Second-order Differential Equations", *International Journal of Computational Mathematics*, **77,** 117-124.

Awoyemi, D. O. (2005) Algorithmic Collocation Approach for Direct Solution of Fourth-order Initial Value problems of ODEs", *International Journal of Computational Mathematics*, **82,** 271-284.

Butcher, J. C. (2003). *Numerical Methods for Ordinary Differential Equations.* West Sussex: John Wiley & Sons Ltd.

Chu. M. T. & Hamilton, H. (1987). Parellel Solution of Ordinary Differential Equations by Multiblock Methods. SIAM *Journal of Scientific and Statistical Computation.* **8,** 342-553.

Fatunla, S. O. (1988). *Numerical Methods for Initial Values Problems in Ordinary Differential Equations.* New York: Academic Press.

Fatunla, S. O. (1994). Block Methods for Second Order Initial Value Problems. *International Journal of Computer Mathematics*, **41,** 55-63.

Ismail, F., Ken, Y. L. & Othman, M. (2009). Explicit and Implicit 3-point Block Methods for Solving Special Second Order Ordinary Differential Equations Directly.

International Journal of Math. Analysis, **3**(5), 239-254.

Jator, S. N. and Li, J. (2009). A Self-Starting Linear Multistep Method for a Direct Solution of the General Second-Order Initial Value Problem. *International Journal of Computer Mathematics,* **86**(5), 827-836.

Kayode, S. J. (2004). *A Class of Maximal Order Linear Multistep Collocation Methods for Direct Solution of Ordinary Differential Equations.* Unpublished doctoral dissertation, Federal University of Technology, Akure, Nigeria.

Kayode, S. J. (2010). A zero-stable Optimal Method for Direct Solution of Second-order differential Equation", *Journal of Mathematics and Statistics,* **6**, 367-371.

Lambert, J. D. (1973). *Computational Methods in* ODEs. New York: John Wiley.

Simo, T. E. (2002). Dissipative Trigonometrically-fitted Methods for Secod Order IVPs Oscillating Solutions. *Int. J. Mod. Phys,* **13**(10), 1333-1345.