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# Mathematical Model of Oxygen Concentration on County Solid Waste Management at Dumpsite

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## Abstract

This study presents the development of one-dimensional mathematical model capable of simulating simultaneous processes of oxygen flow. The resulting governing equations is partial differential equation (PDE) which have been solved by separation of variables method. The goal is to study the three transport parameters; effective diffusivity, decay constant rate and porosity on the oxygen concentration which results in degradation of refuse because after a long-term process in the dumpsite. Solutions of the model equation are obtained using Separation of variables. The results are presented graphically. From the simulated results it is found that for the particular time, oxygen concentration decreases with increase in reaction rate constant (k) with times and depths at the dumpsite. Oxygen concentration at the dumpsite increases with increase in porosity at a particular time at all depths. Oxygen concentration increases with increase in effective diffusivity (ϕ) time and depth.

**Keywords:** decay constant, diffusivity, porosity, Partial differential equations

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## Introduction

Over the past decade, industrialized nations have increasingly recognized the significant consequences of waste generated by both industrial and non-industrial activities. Modern societies produce large quantities of waste, which can be classified as either solid or hazardous. These wastes originate from various sources, including agriculture, manufacturing industries, mining, and milling operations. In some instances, waste is also associated with by-products from gas recovery systems at landfill sites, although such systems are primarily intended to mitigate environmental harm. The growing concern over excessive material consumption has led to efforts aimed at waste minimization and more efficient disposal practices (UNEP, 2024).

Waste generation is an inevitable consequence of human activity, driven by factors such as economic development, urbanization, and rising living standards. As cities grow, both the volume and complexity of solid waste increase, exacerbating environmental and public health challenges (UNEP, 2024). When waste is not effectively managed, it leads to urban environmental degradation, places additional pressure on natural resources, and heightens the risk of disease. This problem is particularly acute in developing countries, where basic services—including water supply, sanitation, transportation infrastructure, and waste collection—are often inadequate (UN-Habitat, 2020).

Solid waste management encompasses waste segregation at the source, storage, collection, transportation, and final disposal in an environmentally responsible manner (UNEP, 2024). It remains a critical public health service, as inadequate waste handling contributes to the spread of diseases such as cholera, typhoid, diarrhea, and malaria

(WHO and UNEP, 2008). In both developed and developing countries, municipal governments are typically responsible for waste collection and disposal, either directly or through contracted private operators. Despite the considerable financial investments—developing nations often allocate 20–50% of their municipal budgets to solid waste management—many still struggle to manage the increasing waste volumes driven by rapid urbanization and population growth (World Bank, 2022). In a recent assessment by the World Health Organization, poor waste management continues to rank among the most urgent environmental health issues in low-income regions, second only to unsafe water (WHO and UNEP, 2008).

Solid waste is a broad term encompassing all discarded materials, excluding hazardous waste, liquid waste, and air emissions. It is generally classified into three main categories: municipal, industrial, and agricultural waste (Ghosh and Ghosh, 2022). Municipal solid waste arises from residential, commercial, institutional, construction, and demolition activities, while industrial solid waste is generated from manufacturing and service-related operations (Li *et al*., 2023). Agricultural waste, including crop residues and animal manure, is often repurposed as fertilizer or soil conditioner (UNEP, 2024).

A comprehensive solid waste management system integrates waste reduction, recycling, composting, incineration, and landfill disposal to minimize environmental harm (WHO and UNEP, 2008). County-level waste strategies emphasize waste handling from the point of generation to final disposal (Kaza *et al*., 2018). The overarching goal is to protect the environment, enhance public health, and promote sustainability. Achieving this requires waste minimization, effective segregation, reliable collection, recycling, and responsible disposal practices (Laner *et al*., 2012). Waste degradation in dumpsites spans several stages involving chemical and biological reactions, including the transition, acid, methane fermentation, and maturation phases (EPA, 2023). These processes influence landfill gas and leachate generation, contributing to pollution (Saqib and Bäckström, 2024).

Municipal solid waste is a persistent source of emissions, especially leachate and biogas. Environmental conditions in dumpsites—driven by rainwater infiltration and oxygen/carbon dioxide diffusion—create aerobic and acidic environments, increasing the mobility of heavy metals (Moqsud *et al*., 2023). Continuous monitoring of closed dumpsites is necessary to assess impacts on groundwater, surface water, soil, and air quality (Laner *et al*., 2012). Hazardous waste remains particularly dangerous due to its toxicity and long environmental persistence. Modern disposal technologies include incineration, sanitary landfills, and deep-mine deposition, with sanitary landfills now preferred for their hygienic and cost-effective operation (Tan *et al*., 2022). Despite improvements, groundwater contamination from landfill leachate is still reported. Recent studies show leachate contains high levels of heavy metals In the early stages of waste decomposition, leachate often contains high concentrations of heavy metals, organic matter (e.g., 2,700–4,300 mg/L TOC), and low pH levels (5–6) (Saqib and Bäckström, 2024; Moqsud *et al.*, 2023. Monitoring indicators such as pH, redox potential, and chemical oxygen demand remains crucial for environmental safety (Bilgili *et al*., 2024).

## Formulation of the Model

The study presents the assumption of the Model equation, a one-dimensional partial differential mathematical model for oxygen concentration, and its method of solution.

### *Assumptions of the Model*

The assumptions include;

i. The porous medium through which migration takes place is homogeneous, isotropic, and saturated.
ii. Porosity of the dumpsite liner material ($\epsilon$), bulk density ($\rho dry$), Darcy's velocity ($v$), and dispersion coefficient ($D$), do not vary with period of migration and depth along the dumpsite.
iii. The model equation is a partial differential equation of one-dimensional type.

iv. Dumpsite is assumed to behave as a bioreactor.
v. The direction of the flow of oxygen is vertically downward.
vi. Gases can be transported by flow and by molecular diffusion into and out of the dumpsite.

### *The Model equation*

Let's $\frac{\phi\partial^2 C}{\partial x^2}$ - Represents the dispersion of the solute (oxygen concentration, $C$ ) due to molecular diffusion in the porous medium. The dispersion coefficient $\phi$ describes the rate at which the solute spreads out due to diffusion. $-V\frac{\partial C}{\partial x}$- Represents the advection of the solute due to Darcy's velocity $V$. Advection refers to the movement of the solute with the bulk flow of the fluid in the porous medium. The negative sign indicates that the solute is being transported in the direction opposite to the direction of increasing $x$. In other words, if $V$ is positive (indicating flow in the positive $x$direction), then the solute is being transported against the flow direction. $-\varepsilon\frac{\partial C}{\partial t}$- Represents the change in solute concentration over time $t$ due to processes such as adsorption or decay. The porosity $\varepsilon$ of the dumpsite liner material accounts for the volume fraction of void space in the porous medium. The negative sign indicates that the concentration of the solute is decreasing with time. $-kC$ - Represents the rate of change of solute concentration due to a reaction process with a rate constant $k$. This represents processes such as chemical reactions or biological transformations. The negative sign indicates that the concentration of the solute is decreasing due to the reaction processes.

Putting the above expressions together give the model equation below;

$$\frac{\phi\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} - \varepsilon\frac{\partial C}{\partial t} - kC = 0 \qquad (1)$$

where $k$ is *reaction rate constant*

**Solution to the Model**

The above equation can be solved by the separation of variables method. The basic idea is to convert the second order partial differential equation into two ordinary differential equations. First, we assume that the solution equals the product X(x) T(t) and direct substitution into the partial differential equation yields two ordinary differential equations.

Assume that

$$\mathrm{C}\,(\mathrm{X.T}) = \mathrm{X(x)\,T(t)} \qquad (2)$$

Thus

$$\left.\begin{aligned} \mathrm{C}_x &= \mathrm{X}'(\mathrm{x})\mathrm{T(t)} \\ \mathrm{C}_{xx} &= \mathrm{X}''(\mathrm{x})\,\mathrm{T(t)} \\ \mathrm{C}_t &= \mathrm{X(x)T}'(\mathrm{t}) \end{aligned}\right\} \qquad (3)$$

Substituting (2) and (3) into the equation (1), we have,

$$\phi\mathrm{X}''(\mathrm{x})\,\mathrm{T(t)} - V\mathrm{X}'(\mathrm{x})\mathrm{T(t)} - \varepsilon\mathrm{X(x)T}'(\mathrm{t}) - \kappa\mathrm{X(x)\,T(t)} = 0$$

Dividing the entire equation by X(x) T(t).

$$\frac{\phi\,\mathrm{X}''(\mathrm{x})\,\mathrm{T(t)}}{\mathrm{X(x)\,T(t).}} - \frac{\mathrm{V\,X}'(\mathrm{x})\,\mathrm{T(t).}}{\mathrm{X(x)\,T(t).}} - \varepsilon\frac{\mathrm{X(x)\,T}'(\mathrm{t})}{\mathrm{X(x)\,T(t).}} - \kappa\frac{\mathrm{X(x)\,T(t).}}{\mathrm{X(x)\,T(t).}} = 0$$

$$\phi\frac{\mathrm{X}''}{\mathrm{X}} - V\frac{\mathrm{X}'}{\mathrm{X}} - \varepsilon\frac{\mathrm{T}'}{\mathrm{T}} - \kappa = 0$$

$$\phi\frac{\mathrm{X}''}{\mathrm{X}} - V\frac{\mathrm{X}'}{\mathrm{X}} - \kappa = \varepsilon\frac{\mathrm{T}'}{\mathrm{T}}$$

The left-hand side is a function of x only, and the right-hand side is a function of t, both sides must be equal to a constant, say $-\lambda^2$

$$\frac{\phi\,\mathrm{X}''}{\mathrm{X}} - \frac{\mathrm{V\,X}'}{\mathrm{X}} - \kappa = \varepsilon\frac{\mathrm{T}'}{\mathrm{T}} = -\lambda^2$$

Therefore

$$\frac{\phi\,\mathrm{X}''}{\mathrm{X}} - \frac{\mathrm{V\,X}'}{\mathrm{X}} - \kappa = -\lambda^2 \qquad (4)$$

Similarly;

$$\varepsilon\frac{\mathrm{T}'}{\mathrm{T}} = -\lambda^2 \qquad (5)$$

Now the two ordinary differential equations (4) and (5) can be solved separately. The solution to equation (4) for X(x) involves finding the roots of the characteristic's equation. The solution to the equation (5) for T (t) is a standard exponential function.

Thus,

$$\frac{\phi\,\mathrm{X}''}{\mathrm{X}} - \frac{\mathrm{V\,X}'}{\mathrm{X}} - \kappa = -\lambda^2$$

$$\phi\mathrm{X}'' - \mathrm{V\,X}' - X\kappa + \lambda^2 X = 0$$

Assume a solution of the form $\mathrm{X(x)} = e^{rx}$

$$\phi\mathrm{r}^2 e^{rx} - \mathrm{V\,r}e^{rx} - \kappa e^{rx} + \lambda^2 e^{rx} = 0$$

Dividing through by $e^{rx}$, we have

$$\phi\mathrm{r}^2 - \mathrm{V\,r} - \kappa + \lambda^2 = 0$$

Using the formula method of solving quadratic equations, that is $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

where , $a = \phi, b = -\mathrm{V}, \mathrm{c} = -\kappa + \lambda^2$

Therefore, the discriminant (Δ) given by

$V^2 - 4\phi(-\kappa + \lambda^2)$,

gives the nature of the roots.

If $\Delta > 0$: then the equation has two real and distinct roots.

If Δ=0: the equation has repeated roots.

If Δ,<0: the equation has complex conjugate roots.

Let's consider the case where $\Delta > 0$

$$r = \frac{V \pm \sqrt{v^2 - 4\phi(-\kappa + \lambda^2)}}{2\phi}$$

$$r_1 = \frac{V + \sqrt{v^2 - 4\phi(-\kappa + \lambda^2)}}{2\phi}$$

$$r_2 = \frac{V - \sqrt{v^2 - 4\phi(-\kappa + \lambda^2)}}{2\phi}$$

The general solution for X (x) can be expressed as a linear combination of the roots.

$$X(x) = Ae^{\alpha x} + Be^{\beta x} \tag{6}$$

Where A and B are constants that can be determined by initial or boundary conditions and $r_1 = \alpha, r_1 = \beta$

Similarly;

$$\varepsilon \frac{T'}{T} = -\lambda^2$$

Dividing through by $\varepsilon$

$$\frac{T'}{T} = \frac{-\lambda^2}{\varepsilon}$$

$$\frac{dT}{dt} = \frac{-\lambda^2}{\varepsilon} T$$

By separation of variables, we have,

$$\frac{dT}{T} = \frac{-\lambda^2}{\varepsilon} dt$$

Integrating both sides

$$lnT = \frac{-\lambda^2}{\varepsilon} t + c$$

Taking exponential of both sides, we have,

$$T(t) = e^{\frac{-\lambda^2}{\varepsilon} tc}$$

Let $e^c = C_1$

$$T(t) = C_1 e^{\frac{-\lambda^2}{\varepsilon} t} \tag{7}$$

Where $C_1$ is a constant.

The overall solution to the partial differential equation is the product of X(x) T(t).

Therefore

$$C(x,t) = \left(Ae^{\alpha x} + Be^{\beta x}\right) C_1 e^{\frac{-\lambda^2}{\varepsilon} t}$$

$$C(x,t) = AC_1 e^{\alpha x} e^{\frac{-\lambda^2}{\varepsilon} t} + BC_1 e^{\beta x} e^{\frac{-\lambda^2}{\varepsilon} t}$$

Let $AC_1 = P$ and $BC_1 = Q$, then

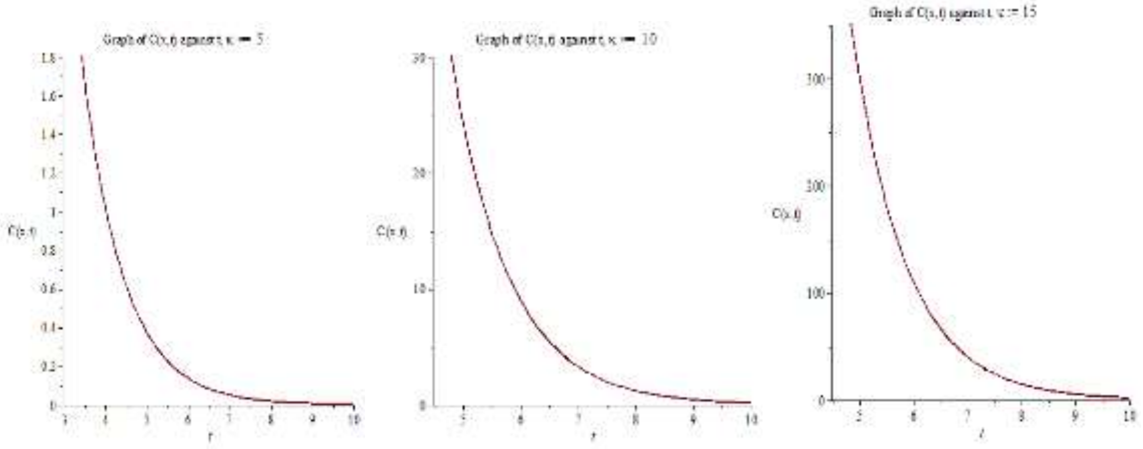
$$C(x,t) = Pe^{\alpha x - \frac{\lambda^2}{\varepsilon} t} + Qe^{\beta x - \frac{\lambda^2}{\varepsilon} t} \tag{8}$$

## Results and Discussion

From the model solution of (8), we looked at the results on the effects of the rate of decay constant, the effect of effective diffusivities and the effect of porosity with respect to time and depth on oxygen concentration in the dumpsite.

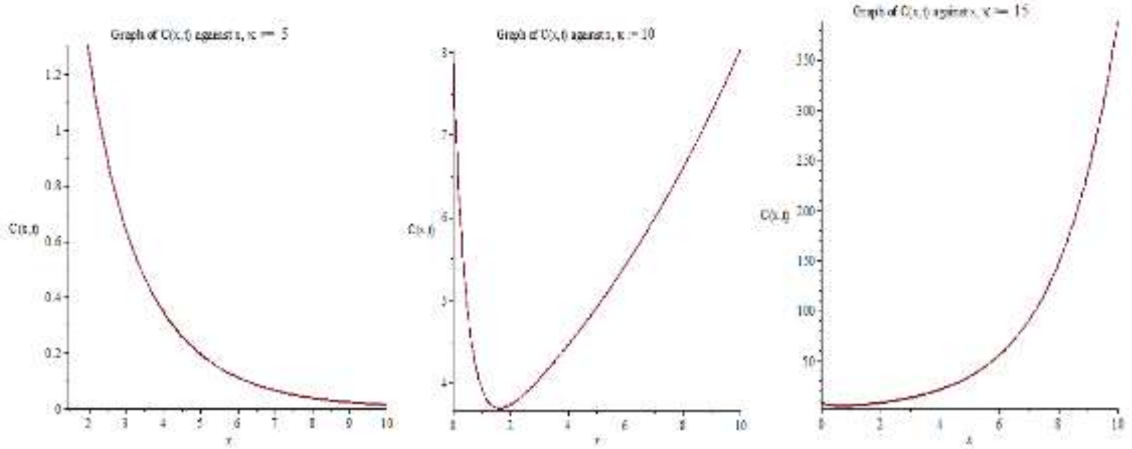
### *Effect of Rate of Decay Constant*

From equation (8), with the use of Maple software we get the results of the effects of a rate of decay constant. The results are represented graphically as seen in figure 1a and b below.



*1a* *1b* *1c*

Figure 1. Graph of oxygen concentration against time at varying rate of decay κ



2a 2b 2c

Figure 2. Graph of oxygen concentration against depth at varying rate of decay

The effect of a decay rate constant with respect to time and depth on oxygen concentration in the dumpsite is shown in Figures 1 and 2 respectively. As we have observed, the oxygen concentration decreases with time and decreases with respect to depth, and increases exponentially as decay constant increases shown in Figures 2(b) and 2(c) respectively. The observed trends in oxygen concentration can be explained by several factors related to oxygen consumption, diffusion, and production:

***Decrease in Oxygen Concentration with an Increase in the Rate Constant over Time***

The rate constant ($k$) represents the reaction rate of oxygen consumption, likely due to biological (e.g., microbial respiration) or chemical (e.g., oxidation reactions) processes. As $k$ increases, the rate of oxygen depletion increases, leading to a faster decline in oxygen concentration over time. This suggests that higher reaction rates accelerate oxygen consumption, reducing its availability in the system.

***Decrease in Oxygen Concentration with Depth***

Oxygen diffusion from the surface is a key process controlling its distribution in a system like water bodies or soil layers. At greater depths, oxygen is less available due to reduced diffusion from the surface and increased consumption by organisms and chemical reactions. Microbial activity and decomposition processes further contribute to oxygen depletion at deeper levels.

***Exponential Increase in Oxygen Concentration***

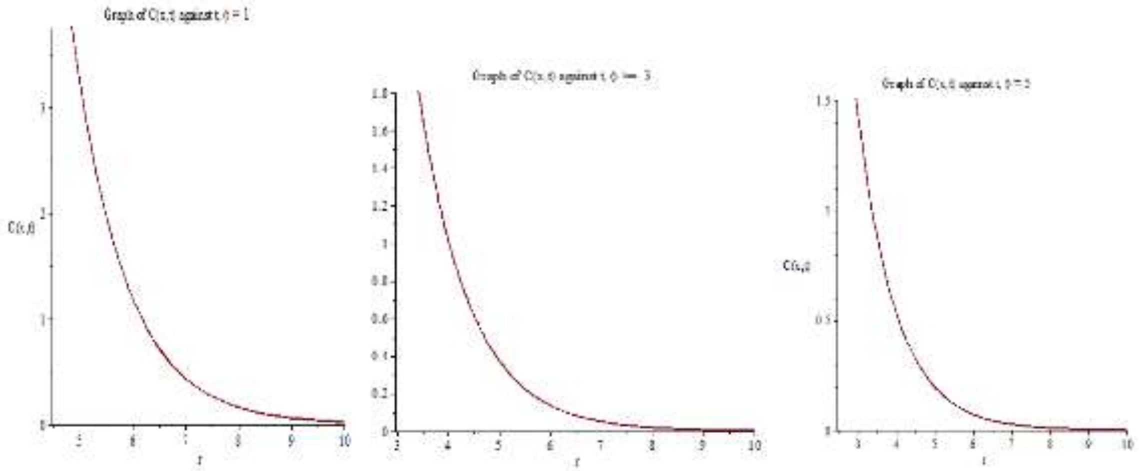
This behavior suggests a recovery or replenishment mechanism at play, possibly due to:

Photosynthesis in aquatic environments, where oxygen is generated by algae or aquatic plants. Oxygen diffusion from an external source, such as atmospheric absorption at the water surface. Turbulent mixing that redistributes oxygen in certain conditions.

The exponential increase could indicate that oxygen replenishment follows a natural recovery process, possibly governed by diffusion-limited kinetics or biological growth.

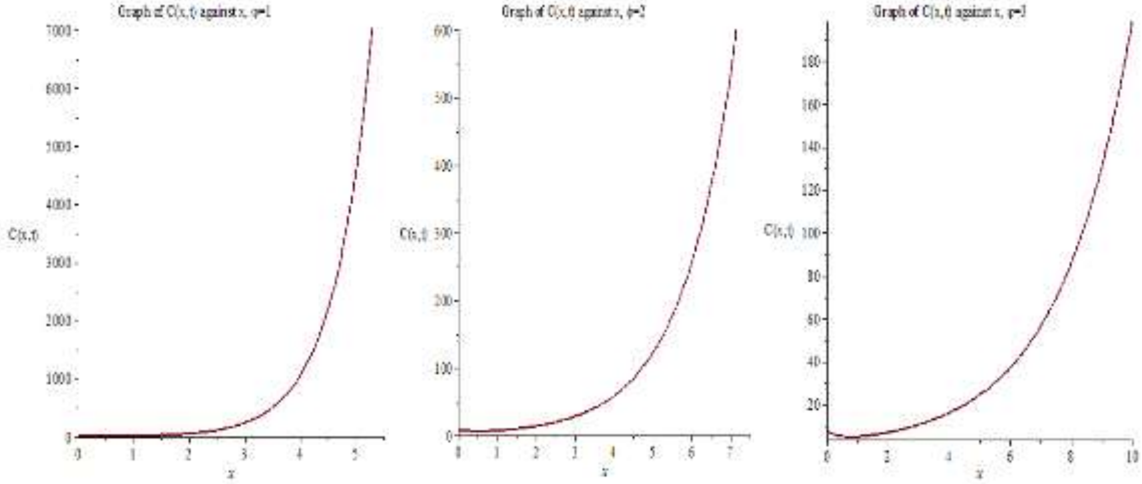
***Effect of Effective Diffusivity***

The effect of effective diffusivities with respect to time and depth on oxygen concentration at the dumpsite, the results are presented graphically as seen in figure 2a and 2b below.



*3a* *3b* *3c*

Figure3. Graph of oxygen concentration against time at varying diffusivity coefficient ф



4a 4b 4c

Figure 4. Graph of oxygen concentration against depth at varying diffusivity coefficient ф

The figures show that, the oxygen concentration decreases with time and increases exponentially with depth as effective diffusivity increases, as shown in Figures (a) and (b). The results indicate the following environmental and physical implications for oxygen concentration at the dumpsite:

***Decreasing Oxygen Concentration Over Time***

This suggests that as time progresses, the oxygen available in the dumpsite reduces. Possible reasons include microbial decomposition of organic waste, which consumes oxygen; Anaerobic conditions may develop over time, leading to the production of gases like methane ($CH_4$) and carbon dioxide ($CO_2$) instead of sustaining aerobic degradation; and this could impact waste stabilization and contribute to greenhouse gas emissions.

***Increasing Oxygen Concentration with Depth in Effective Diffusivity***

If effective diffusivity increases with depth, it means oxygen is being transported more efficiently downward. This could be due to improved porosity or permeability of waste materials, facilitating better gas exchange. It suggests that deeper layers may receive more oxygen, potentially delaying the onset of anaerobic conditions in lower regions. This can enhance aerobic microbial activity in deeper layers, promoting faster decomposition of organic waste.

***Effect of Porosity***

Result in Figure 5 and 6 shows that oxygen concentration decreases, as time increases and then becomes a constant at particular depth.

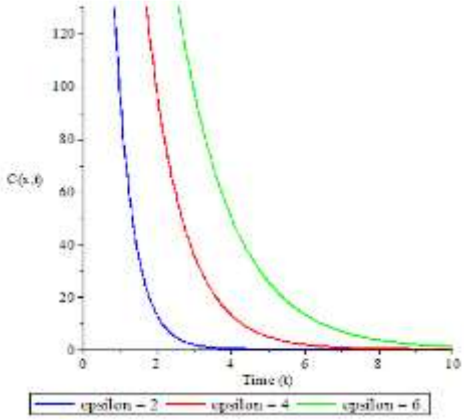


Figure 5. Graph of oxygen concentration against time at varying porosity ε

At $t = 0$, all curves start from different initial concentrations due to the exponential terms involving $\alpha x$ and $\beta x$. As time progresses, the concentration decays more slowly for higher ε, meaning greater porosity results in slower depletion of oxygen.

This behavior reflects how porous media with higher ε may allow slower diffusion or attenuation.

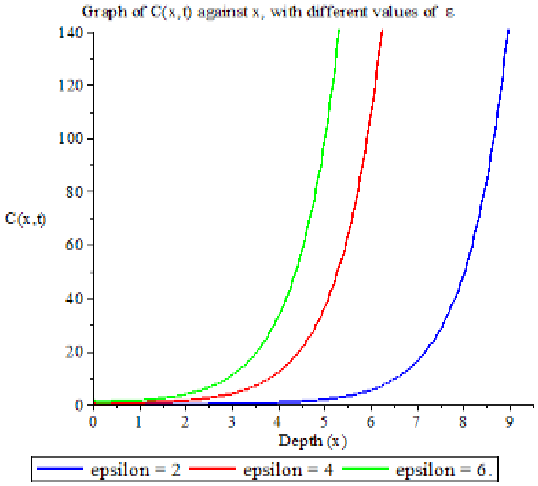


Figure. 6 Graph of oxygen concentration against depth at varying porosity

The graph of oxygen concentration vs. depth for varying porosity $\varepsilon$ at constant time $= t_0$, we use the same equation. The concentration profiles differ in magnitude and shape depending on $\varepsilon$. As porosity increases, the rate at which the exponential decay term affects the concentration decreases, so the curves are broader and decay more slowly across depth.

The results suggest that oxygen concentration decreases with time irrespective of porosity and decreases exponentially with depth, after which it stabilizes. These results suggest the following:

***Oxygen Depletion Over Time***: The continuous decrease in oxygen concentration over time means that microbial and chemical processes at the dumpsite are actively consuming oxygen. This is typical in environments where organic matter is degrading, as microorganisms use oxygen to break down waste.

***Exponential Increase with Depth (Followed by Stabilization)***: Initially, oxygen concentration increases with depth, likely due to lower microbial activity and slower oxygen consumption in deeper layers. However, after a certain depth, the concentration stabilizes, indicating that oxygen diffusion reaches a balance with its consumption or that deeper layers have minimal oxygen demand due to reduced biological activity.

## Conclusion

The simulated results indicate that oxygen concentration at the dumpsite decreases at the reaction rate constant (k) over time and depth. Also, the oxygen concentration decreases with time and increases exponentially with depth for all values of effective diffusivity. Conversely, oxygen concentration decreases with time and became stable irrespective of the porosity (ε) and oxygen concentration increases exponentially with depths. However, variations in velocity (V) have no significant impact on oxygen concentration, as the low Reynolds number in porous media ensures flow remains within the Darcy's region. This promotes effective aeration and microbial activity in the soil, facilitating rapid refuse degradation.

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